Homework 7 Solutions

4.2 Determinants

49. We use Theorems 4.8, 4.9 and $\det A = 3$, $\det B = -2$ to compute $\det(B^{-1}A)$.

$$\det(B^{-1}A) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det A) \overset{\text{Thm 4.9}}{=} \left(\frac{1}{\det B}\right)(\det A) \overset{\text{givens}}{=} (-\tfrac{1}{2})(3) = -\tfrac{3}{2}$$

55. We use the fact that $A^2 = A \overset{\text{Ex 47}}{\Rightarrow} (\det A)^2 = \det A$ to find all possible values of $\det A$.

$$(\det A)^2 = \det A \Rightarrow (\det A)^2 - \det A = 0 \Rightarrow \det A(\det A - 1) = 0 \Rightarrow \det A = 0, 1$$

Q: If we let $x = \det A$, what does the above calculation look like?

A: With $x = \det A$: $x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$.
This clarifies the algebra and helps us remember that $\det A$ is a scalar.

4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices

1. We follow the procedure outlined before Example 1.

(a) The characteristic polynomial is $\det(A - \lambda I) = 0$, so we have:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 3(-2) = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4).$$

(b) The characteristic equation is $(\lambda-3)(\lambda-4) = 0$, which has solutions $\lambda_1 = 3$ and $\lambda_2 = 4$.

(c) To find the eigenvectors corresponding to λ_1, we find the null space of $A - 3I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$.

Row reduction produces $\left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$.

Thus, $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in the eigenspace E_3 if and only if $x_1 - \frac{3}{2}x_2 = 0 \Leftrightarrow x_2 = \frac{2}{3}x_1$.

Thus, $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}\right)$.

Similarly, $A - 4I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(d) Each eigenvalue has algebraic and geometric multiplicity 1.

4. (a) $$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(1-\lambda)-(1-\lambda)-\lambda(1-\lambda)^2$$
$$= -2(\lambda-1)-\lambda(\lambda-1)^2 = (\lambda-1)(\lambda-2)(\lambda+1).$$

(b) $(\lambda-1)(\lambda-2)(\lambda+1)=0 \Leftrightarrow \lambda_1=-1,\ \lambda_2=1,\ \lambda_3=2.$

(c) $$A+I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right).$$

$$A-I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right).$$

$$A-2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

(d) Each eigenvalue has algebraic and geometric multiplicity 1.

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9. (a) $$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 4 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix}$$
$$= [(3-\lambda)(1-\lambda)+1]\left[(\lambda-1)^2-4\right] = (\lambda-2)^2(\lambda-3)(\lambda+1).$$

(b) $(\lambda-2)^2(\lambda-3)(\lambda+1)=0 \Leftrightarrow \lambda_1=-1,\ \lambda_2=\lambda_3=2,\ \lambda_4=3.$

(c) $$A+I = \begin{bmatrix} 4 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{-1} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix}\right).$$

$$A-2I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right).$$

$$A-3I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}\right).$$

(d) -1 and 3 have algebraic and geometric multiplicity 1,
while 2 has algebraic multiplicity 2 and geometric multiplicity 1.

20. Given $A^n = O$, we need to show if $A\mathbf{x} = \lambda\mathbf{x}$ then $\lambda = 0$. That is:
1) 0 is an eigenvalue of A and 2) if λ is an eigenvalue of A, then $\lambda = 0$.

To prove assertion 1 we make the following observation:

Q: What is the contrapositive of Theorem 4.16?
A: A is *not* invertible if and only if 0 *is* an eigenvalue of A.

Q: What does the contrapositive of Theorem 4.16 imply?
A: $\det A = 0$ if and only if 0 *is* an eigenvalue of A. Why? Because the contrapositive of Theorem 4.6 in Section 4.2 implies $\det A = 0$ if and only if A is *not* invertible.

So, to prove 0 is an eigenvalue of A it suffices to show $\det A = 0$.
That is, if $A^n = O$, then $\det A = 0$. So, 0 is an eigenvalue of A.

Since $(\det A)^n \overset{\substack{\text{Thm 4.8}\\ \text{Sect 4.2}}}{=} \det(A^n) \overset{\substack{A^n=O\\ \text{given}}}{=} \det O = 0$, $\det A = 0$. So, 0 is an eigenvalue of A.

Next we show if λ is an eigenvalue of A, then $\lambda = 0$.

If $A\mathbf{x} = \lambda\mathbf{x}$, then Theorem 4.18(c) implies $A^n\mathbf{x} = \lambda^n\mathbf{x} = O\mathbf{x} = \mathbf{0}$.
Since $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$. So $\lambda^n\mathbf{x} = \mathbf{0}$ implies $\lambda^n = 0$. Therefore, $\lambda = 0$ as claimed.

25. As noted in Theorem 4.17(d), $A \longrightarrow I$ if and only if A is invertible.

Q: If the conjecture of this exercise were true, what would that imply?
A: Since the only eigenvalue of I is 1, all invertible matrices would only have eigenvalue 1. This is clearly nonsense. However, they may be *related*.

Q: Let $\mathbf{x}$ be an eigenvector of A corresponding to eigenvalue λ.
If $A \xrightarrow{R_i \leftrightarrow R_j} B$, that is $B = E_{ij}A$, what goes wrong?
A: Since $B = E_{ij}A$, we have $B\mathbf{x} = E_{ij}(A\mathbf{x}) = \lambda(E_{ij}\mathbf{x})$.
So the components of $\mathbf{x}$ are interchanged and $\mathbf{x}$ fails to be an eigenvector for B.

Q: If $A \xrightarrow{kR_i} B$, what goes wrong?
Q: If $A \xrightarrow{R_i + kR_j} B$, what goes wrong?

Q: So, if $A \longrightarrow B$, we have seen their eigenvalues are not necessarily equal. However:
If $A \longrightarrow B$, is there a relationship among the eigenvalues and eigenvectors?
A: Hint: $2I$ has eigenvalue 2. Can this process be generalized? See Exercise 41.

28. (a) $C(p) = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix}$.

So, the characteristic polynomial of $C(p)$ is $\begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \lambda a + b$.

(b) Suppose that λ is an eigenvalue of $C(p)$, with eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C(p) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ax_1 - bx_2 \\ x_1 \end{bmatrix} \Leftrightarrow$

$x_1 = \lambda x_2$, so $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

4.4 Similarity and Diagonalization

2. $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 16$, while

$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16$. Thus, by Theorem 4.22(d), $A \nsim B$.

4. $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 2\lambda$, while

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$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 3\lambda \Rightarrow A \nsim B$.

10. A has eigenvalue 3 with algebraic multiplicity 3.

Furthermore $A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so this eigenvalue has geometric multiplicity 1,

and thus A is not diagonalizable, by the Diagonalization Theorem.

11. Expanding along the first row,

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(-\lambda)-1\right] - (1-\lambda)$$

$$= (1-\lambda)\left(\lambda^2-\lambda-2\right) = -(\lambda+1)(\lambda-1)(\lambda-2).$$

So, A has eigenvalues -1, 1, and 2, and Theorem 4.25 tells us that that A is diagonalizable.

We find bases for the eigenspaces:

$A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$, so E_{-1} has basis $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so E_1 has basis $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so E_2 has basis $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ satisfy $P^{-1}AP = D$.